# Note for the random modulo $q$ model

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This note is organized as follows: Sec. I recalls the graphical representation of the Potts model. With the preliminaries, we construct the random modulo $q$ model in Sec. II, which is equivalent to the graphical representation in Sec. I with parameter $p = (1-e^{-K})/q$. Connections between the random modulo $q$ model and the random cluster model are proposed, providing a brand-new algorithm for simulating the $q$-state Potts model with the integer $q$.

## I. GRAPHICAL REPRESENTATION FOR THE POTTS MODEL

For the ferromagnetic Potts model without external field, the Hamiltonian reads

$$-\frac{\mathcal{H}}{T} = \sum_{b=\langle i,j\rangle} K\delta(\sigma_i - \sigma_j)\ , \tag{1}$$

where

$$\delta(n) = \begin{cases} 1 & ,n=0 \\ 0 & ,\text{otherwise} \end{cases}\ , \tag{2}$$

$K > 0$ is the dimensionless coupling strength, $T$ is the temperature and $\sigma_i \in \{0, 1, \cdots, q-1\}$. We use $b$ to denote the bond between nearest-neighbor sites $i$ and $j$. Another notation $(i, \nu)$ is also used to label bond by specifying the direction $\nu$ of nearest neighbor relative to site $i$. The partition function writes

$$\mathcal{Z} = \sum_{\{\sigma\}} \prod_{b=\langle i,j\rangle} e^{K\delta(\sigma_i-\sigma_j)}\ , \tag{3}$$

where the summation for spin variables runs over all lattice sites.

To reformulate the partition function in graphical representation, we utilize the Discrete Fourier Transform (DFT) and Inverse Discrete Fourier Transform:

$$F(n) = \sum_{\sigma=0}^{q-1} f(\sigma) e^{-j\frac{2\pi}{q}\sigma n} \tag{4}$$

$$f(\sigma) = \frac{1}{q}\sum_{n=0}^{q-1} F(n) e^{j\frac{2\pi}{q}\sigma n}\ . \tag{5}$$

Then, the Boltzmann statistical weight $f(|\sigma_i - \sigma_j|) = \exp(K\delta(\sigma_i - \sigma_j))$ can be expanded to

$$\exp(K\delta(\sigma_i - \sigma_j)) = \frac{1}{q}\sum_{n_b=0}^{q-1} F(n_b) e^{j\frac{2\pi}{q}(\sigma_i-\sigma_j)n_b}\ , \tag{6}$$

where

$$F(n) = \begin{cases} e^K + q - 1 & ,n=0 \\ e^K - 1 & ,\text{otherwise} \end{cases}\ . \tag{7}$$

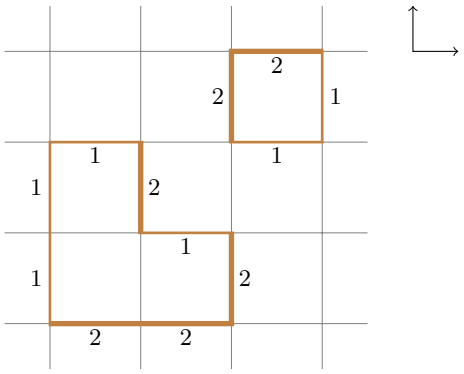


FIG. 1: Example about the configuration of the 3-state Potts model in graphical representation. Positive directions are shown by black arrows, similarly hereinafter.

And the partition function is reformulated to

$$\mathcal{Z} = \sum_{\{n_b\}} \left(\prod_b F(n_b)\right)\left(\prod_i \sum_{\sigma_i=0}^{q-1} e^{j\frac{2\pi}{q}(\sum_\nu \mathrm{sgn}(\nu)n_{i,\nu})\sigma_i}\right)\ , \tag{8}$$

where the trivial factor has been dropped and bond variable $n_b \in \{0, 1, \cdots, q-1\}$. In addition, it's crucial to notice that we have to specify the positive direction for each bond, reflected by $\mathrm{sgn}(\nu)$. Graphically, each bond variable can be represented by $n_b$ edges ascribed to the corresponding bond. Then symmetry of $\sum_{\sigma_i=0}^{q-1} e^{j\frac{2\pi}{q}(\sum_\nu \mathrm{sgn}(\nu)n_{i,\nu})\sigma_i}$ requires $\sum_\nu \mathrm{sgn}(\nu)n_{i,\nu}$ equals zero (modulo $q$) to make non-zero contribution.

Eventually, $\mathcal{Z}$ is reformulated in the graphical representation:

$$\mathcal{Z} = \sum_{\{n_b\}} \left(\prod_i Q(\sum_\nu \mathrm{sgn}(\nu)n_{i,\nu})\right)\left(\prod_b F(n_b)\right)\ , \tag{9}$$

where constraints are expressed explicitly in the first production with the help of function $Q$:

$$Q(n) = \begin{cases} 1 & ,n = 0 \bmod q \\ 0 & ,\text{otherwise} \end{cases}\ . \tag{10}$$

Consider $n_b$ as some kind of "current" ascribed to bond $b$, then the configuration space consists of all (modulo $q$) conserved subgraphs. Example about effective configuration can be seen in FIG. 1.

Before going further, there are some intuitive observations:

(i)Because of the non-negative value of $n_b$, currents can only flow towards positive directions. Accordingly, for simplicity, we can represent the configuration by *undirected* subgraphs instead *directed* subgraphs. Positive directions are introduced to construct constraints for these undirected edges;

(ii)According to bond weight (7), there are two kinds of bond variables: $n_b = 0$ and $n_b \neq 0$. Whereas two types of edges (open and close) exist in the random cluster model, we can also classify $n_b$ into two groups according to whether it equals zero or not.

## II. RANDOM MODULO $q$ MODEL

Let $G = (V, E)$ be a *finite undirected simple graph* with vertex set $V$ and edge set $E$. Elements $v \in V$ and $e \in E$ are used to label site and bond of lattice, respectively. Define vector space $\Omega^{\mathrm{RM}} = \{0, 1, ..., q-1\}^E$ to describe graph $G^{\mathrm{RM}} = (V, E^{\mathrm{RM}})$. The conserved cycle space $\mathcal{C}^{\mathrm{RM}}$ is a subspace of $\Omega^{\mathrm{RM}}$ consisting of all $\omega \in \Omega^{\mathrm{RM}}$ so that the conserved law presented above is satisfied. Apparently, $\mathcal{C}^{\mathrm{RM}}$ is a vector space over $\mathbb{Z}_q$. The uniform measure for random modulo $q$ model is

$$\begin{aligned}\rho_p(\omega) &= \frac{1}{Z_{\mathrm{RM}}} \prod_{e\in E} p^{\delta(\omega(e)-1)} p^{\delta(\omega(e)-2)} \cdots p^{\delta(\omega(e)-(q-1))} [1-(q-1)p]^{\delta(\omega(e))} \\ &= \frac{1}{Z_{\mathrm{RM}}} \prod_{e\in E} p^{1-\delta(\omega(e))} [1-(q-1)p]^{\delta(\omega(e))} \quad , \omega \in \mathcal{C}^{\mathrm{RM}} , \end{aligned} \tag{11}$$

where $p \in [0, 1/q]$. It's easy to verify that the random modulo $q$ model is equivalent to the graphical representation (9) with parameter $p = (1 - e^{-K})/q$.

As for the random cluster model, the vector space of $G^{\mathrm{RC}} = G$ reduces to $\Omega^{\mathrm{RC}} = \{0, 1\}^E$. The uniform measure reads

$$\begin{aligned}\phi_{q,p}(\omega) &= \frac{1}{Z_{\mathrm{RC}}} \prod_{e\in E} p^{1-\delta(\omega(e))} (1-p)^{\delta(\omega(e))} q^{k(\omega)} \\ &= \frac{1}{Z_{\mathrm{RC}}} p^{|\eta(\omega)|} (1-p)^{|E\setminus\eta(\omega)|} q^{k(\omega)} \quad , \omega \in \Omega^{\mathrm{RC}}, \end{aligned} \tag{12}$$

where $\eta(\omega) = \{e \in E | \omega(e) = 1\}$ is the set of open edges and $k(\omega)$ is the number of open components on the vertex set $V$. For the $q$-state Potts model, spin representation and random cluster representation are connected by $p = 1 - e^{-K}$.

To find connections between (11) and (12), we define the map between $\Omega^{\mathrm{RM}}$ and $\Omega^{\mathrm{RC}}$:

**Definition 1.** Let $\psi$: $\Omega^{\mathrm{RM}} \to \Omega^{\mathrm{RC}}$ be the map between two vector spaces. For $\omega \in \Omega^{\mathrm{RM}}$ and $\omega' \in \Omega^{\mathrm{RC}}$, $\omega' = \psi(\omega)$ is defined as $\omega'(e) = 1 - \delta(\omega(e))$ for each $e \in E$.

Actually, $\psi$ maps multiple edges in $\Omega^{\mathrm{RM}}$ to single edge in $\Omega^{\mathrm{RC}}$. With the goal of decomposing $\omega \in \Omega^{\mathrm{RM}}$ into some basic elements, we define the *conserved cycle* by the use of *cycle* in graph theory.

**Definition 2.** A conserved cycle $C$ is an element of $\Omega^{\mathrm{RM}}$ which satisfies the following requirements:

(i)$C' = \psi(C)$ is a cycle;

(ii)The current-conserved law is satisfied in $C$.

Conversely, given a cycle $C' \in \Omega^{\mathrm{RC}}$, we can get one trivial conserved cycle by setting $C(e) = 0$ and $q-1$ non-trivial conserved cycles by setting $C(e) = 1, 2, \cdots, q-1$ for an arbitrary edge $e \in C'$, as shown in FIG.2. And we can label them by $C^{(0)}, C^{(1)}, \cdots, C^{(q-1)}$. For vector space $\Omega^{\mathrm{RC}}$, let $\{C'_1, C'_2, \cdots, C'_c\}$ be a maximal set of independent cycles. Then conserved cycles $\{C_1^{(k)}, C_2^{(k)}, \cdots, C_c^{(k)} | k = 0, 1, \cdots, q-1\}$ can be obtained.

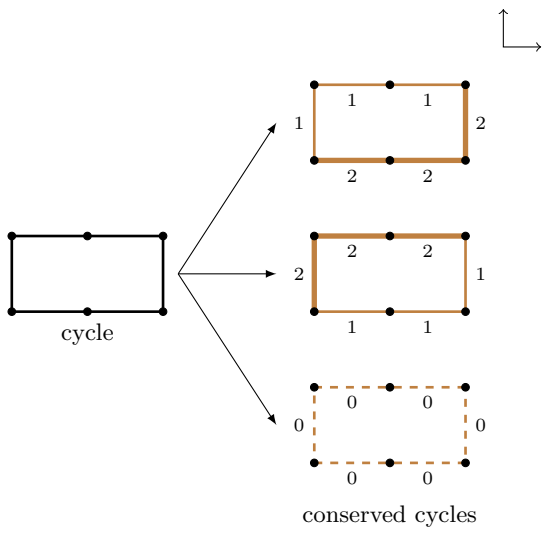


FIG. 2: Let $q$ equals 3. Given a cycle, two non-trivial conserved cycles and one trivial conserved cycle can be constructed.

**Theorem 1.** *$\{C_1^{(1)}, C_2^{(1)}, \cdots, C_c^{(1)}\}$ is a basis of the vector space $\mathcal{C}^{RM}$ over $\mathbb{Z}_q$.*

*Proof.* First, suppose $G = (V, E)$ is a connected graph. Let one of its spanning tree is $T = (V, F)$. For each

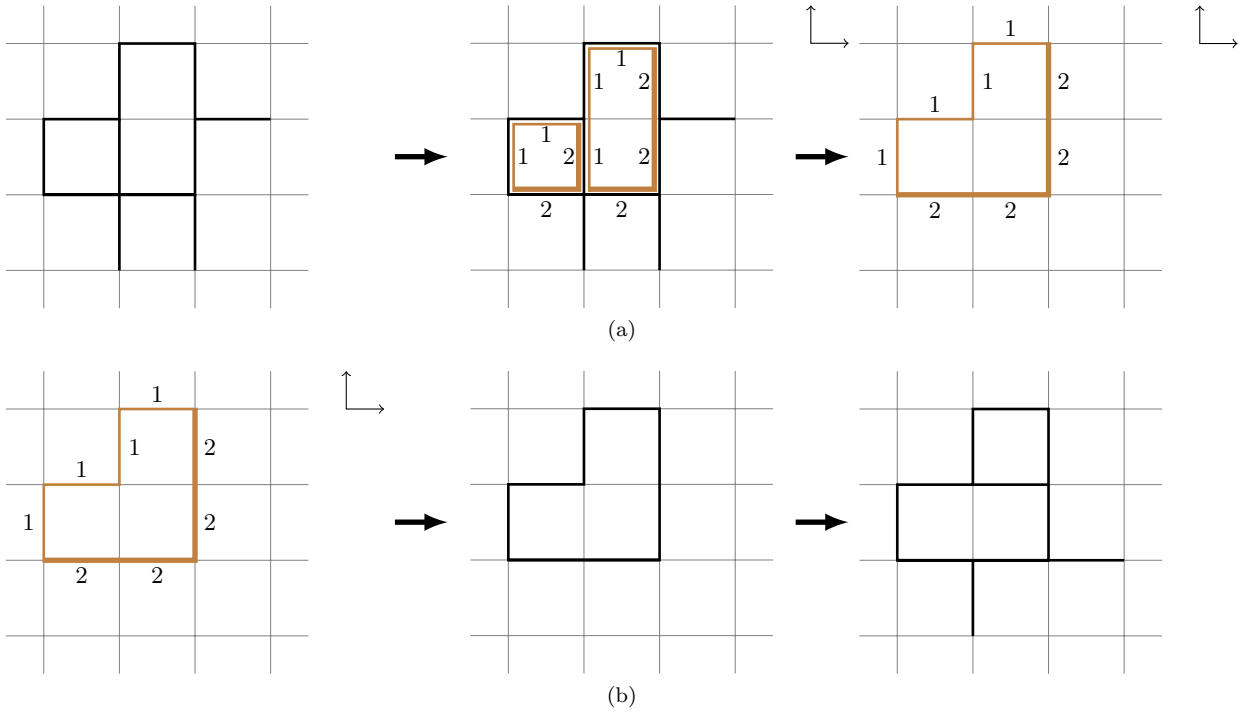


FIG. 3: Transform between two representations. (a) From random cluster representation (RC) to random modulo $q$ representation (RM): Choose a set of independent cycles in RC, construct the conserved cycles and combine these conserved cycles. (b) From RM to RC: Map the configuration in RM to RC, add edges with probability $p/(1-(q-1)p)$ for unoccupied bonds.

$e_k \in E\backslash F$, $1 \le k \le c$, there's only one cycle $C'_{Tk}$ in the vector space of $(V, F\bigcup e_k)$. By this way, we get a maximal set of independent cycles $\{C'_{T1}, C'_{T2}, \cdots, C'_{Tc}\}$ and $e_k$ is owned only by $C'_{Tk}$. Then conserved cycles $\{C^{(1)}_{T1}, C^{(1)}_{T2}, \cdots, C^{(1)}_{Tc}\}$ can be constructed.

Let $\omega \in \mathcal{C}^{\mathrm{RM}}$ and $a_k = \omega(e_k)/C^{(1)}_{Tk}(e_k)$, where the division is carried out over $\mathbb{Z}_q$. Because $C^{(1)}_{Tk}(e_k)$ equals either 1 or $q-1$, $a_k$ is well-defined. Let $\omega_1 = \sum_{k=1}^{c} a_k C^{(1)}_{Tk}$, we proof that $\omega$ equals $\omega_1$.

Because $\omega$ and $\omega_1$ are elements of vector space $\mathcal{C}^{\mathrm{RM}}$. We get $\omega_1 - \omega \in \mathcal{C}^{\mathrm{RM}}$. By definition, $(\omega_1 - \omega)(e) = 0$ for $e \in \{e_k | k = 1, 2, \cdots, c\}$. For $e \in E$ and $e \notin \{e_k | k = 1, 2, \cdots, c\}$, it is an element in the edge set of the spanning tree $T$. And it is impossible to satisfy the conserved law if $(\omega_1 - \omega)(e) \neq 0$. Accordingly, $(\omega_1 - \omega)(e)$ equals 0 for any $e \in E$ and $\omega = \sum_{k=1}^{c} a_k C^{(1)}_{Tk}$. Because $\omega \in \mathcal{C}^{\mathrm{RM}}$ in arbitrary, $\{C^{(1)}_{T1}, C^{(1)}_{T2}, \cdots, C^{(1)}_{Tc}\}$ is a basis of the vector space $\mathcal{C}^{\mathrm{RM}}$. For arbitrary set of independent cycles $\{C^{(1)}_1, C^{(1)}_2, \cdots, C^{(1)}_c\}$, because elements are independent to each other, it is also a basis of the vector space $\mathcal{C}^{\mathrm{RM}}$.

If $G = (V, E)$ is not a connected graph, we can decompose it to $G = \sum_{k=0}^{m} G_k$, $G_i \bigcap G_j = \emptyset$, $0 \le i, j \le m$, where $G_0$ is the set of all isolated vertices and $G_k$ $(1 \le k \le m)$ is a connected graph. For each connected graph $G_k$, we can find a basis $\{C^{(1)}_{k_1}, C^{(1)}_{k_2}, \cdots, C^{(1)}_{k_t}\}$, and their union is a basis of $\mathcal{C}^{\mathrm{RM}}$. □

According to this theorem, any configuration of the random modulo $q$ model can be decomposed to the combination of several conserved cycles. This provides the foundation to build connections between the random modulo $q$ model and the random cluster model.

**Theorem 2.** *Let $p \in [0, 1/q]$ and let $\omega \in G^{RC}$ be a realization of the random cluster model on $G$ with parameters $qp$ and $q$. Let $R = (V, \gamma)$ be a uniform random modulo $q$ subgraph of $(V, \eta(\omega))$, then $R$ is a random modulo $q$ subgraph of $G^{RM}$ with parameter $p$.*

*Proof.* Given a configuration $\omega$ in random cluster model, we first find out all of its independent cycles. Denote the number of these independent cycles by $c(\omega)$. And we can build a basis $\{C^{(1)}_1, C^{(1)}_2, \cdots, C^{(1)}_{c(\omega)}\}$ of the vector space $\Omega^{\omega} = \{0, 1, ..., q-1\}^{\eta(\omega)}$. Then the probability to construct a random modulo $q$ subgraph $(V, g)$ is

$$P(\gamma = g|\omega) = \begin{cases} q^{-c(\omega)} & , \text{if } \psi(g) \subseteq \eta(\omega) \\ 0 & , \text{otherwise} . \end{cases}$$

Denote the vector of $(V, g)$ by $\kappa$. Using the identity that

$c(\omega) = |\eta(\omega)| - |V| + k(\omega)$, we get

$$\begin{aligned} P(\gamma = g) &= \sum_{\omega:\psi(g)\subseteq\eta(\omega)} q^{-c(\omega)}\phi_{qp,q}(\omega) \\ &\propto \sum_{\omega:\psi(g)\subseteq\eta(\omega)} p^{|\eta(\omega)|}(1-qp)^{|E\setminus\eta(\omega)|} \\ &\propto p^{|\psi(g)|}[1-(q-1)p]^{|E\setminus\psi(g)|} \\ &\propto \prod_{e\in E} p^{1-\delta(\kappa(e))}[1-(q-1)p]^{\delta(\kappa(e))}, \kappa \in \mathcal{C}^{\mathrm{RM}}, \end{aligned}$$

which is consistent with Eq. (11). □

**Theorem 3.** *Let $(V, F)$ be a realization of the random modulo $q$ subgraph with parameter $p \in [0, 1/q]$. To each $e \in E\setminus F$, add edge with probability $p_c = \frac{p}{1-(q-1)p}$ on the base of $F$ and get $H$, then graph $(V, H)$ has law $\phi_{qp,q}$.*

*Proof.* For $\omega \in \Omega^{\mathrm{RM}}$, abbreviate $\eta(\psi(\omega))$ to $\eta(\omega)$. For $h \subseteq E$, we have

$$\begin{aligned} P(H = h) &= \sum_{\omega:\eta(\omega)\subseteq h} \rho_p(\omega) p_c^{|h\setminus\eta(\omega)|}(1-p_c)^{|E\setminus h|} \\ &\propto \sum_{\omega:\eta(\omega)\subseteq h} \left(\frac{p}{1-(q-1)p}\right)^{|\eta(\omega)|} \\ &\times \left(\frac{p}{1-(q-1)p}\right)^{|h\setminus\eta(\omega)|} \left(\frac{1-qp}{1-(q-1)p}\right)^{|E\setminus h|} \\ &\propto (qp)^{|h|}(1-qp)^{|E\setminus h|} q^{k(h)} , \end{aligned}$$

consistent with Eq. (12). □

The illustration of the transform between the random modulo $q$ model and the random cluster model is shown in Fig. 3.